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RESEARCH NOTES IN ECONOMICS

A New Approach for Turkish Term Structure: Cubic B-Spline Basis with Variable Roughness Penalty

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Özet: Getiri eğrileri politika yapıcılar ve piyasa oyuncuları açısından önemli bilgi değeri taşımaktadırlar. Genişletilmiş Nelson Siegel (ENS) ve çeşitli düzgünleştirme ceza (smooth penalty) fonksiyonları ile uygulanan kübik B-spline (VRP), en popüler getiri eğrisi hesaplama yöntemlerindendir. Bu notta, Türkiye Hazine tahvilleri için düzgünleştirme cezası ile beraber uygulanan Kübik B-spline yöntemi tahmin edilmekte ve sonuçlar Genişletilmiş Nelson-Siegel yöntemi ile kıyaslanmaktadır. İki yöntem altında da elde edilen çeşitli vadelerdeki getiriler birbirine oldukça yakın seviyededir. Örneklem içi karşılaştırma yapıldığında, VRP yönteminin ENS'ye göre özellikle uzun vadeli getirilerde daha iyi sonuçlar verdiği bulunmuştur. Örneklem dışında ise, iki yöntemin de birbirine oldukça yakın sonuçlar verdiği gözlenmiştir. Ancak, VRP yöntemi getiri eğrisinin kısa ucunda, ENS yöntemi ise getiri eğrisinin uzun ucunda daha iyi örneklem dışı tahmin sağlamaktadır.

Abstract: Yield curves provide critical information to both policy makers and market practitioners. The most commonly used yield curve methodologies are Extended Nelson-Siegel (ENS) and B-spline Basis with various smoothing penalty functions. In this paper, the Cubic B-spline Basis with variable roughness penalty (VRP) methodology is applied on Turkish Treasury bonds and is compared with the ENS methodology. The yield curves constructed under both methods give quite similar results across different maturities. Comparison of in-sample performances indicates that both methods provide similar zero rate curves but in-sample fit of Cubic B-spline Basis with VRP is superior to that of ENS, in particular at the long end of the term structure. Out-of-sample errors of both models are quite close to each other. Nevertheless, out-of-sample performance of the VRP model is superior at the short end of the term structure whereas the ENS dominates at the long end.

1. Introduction

Yield curve describes the relationship between interest rates and time to maturity. It is one of the most fundamental concepts in finance, since valuable information is embedded in the yield curve such as the market's expectations about the future path of interest rates, rich/cheap analysis or asset pricing. However, yield curves are not directly observable although the prices and the yields of bonds are available in the market. To construct a yield curve directly from the market data, one requires zero coupon bonds across a continuum of maturities. However, the bonds are issued across a finite set of maturities. Additionally, most of the bonds for longer term maturities are fixed coupon bonds. Therefore, attempts to extract yield curve from bond market require modeling zero rates or forward rates.

In this respect, models to extract zero rates from the market data can be categorized into two broad categories: parametric and spline-based approaches. Parametric models describe the instantaneous forward rate or zero rate at all maturities as a single function of specific unknown parameters. One of the most popular parametric models is proposed by Nelson and Siegel (1987) and later extended by Svensson (1994). Spline based models fit a yield curve to the bond data by relying on a piecewise polynomial for different maturity segments. McCulloch (1971), one of the earliest papers in spline based models, introduces quadratic splines to fit a yield curve. Later, Vasicek and Fong (1982) use exponential splines. However, these splines suffer from numerical instability. Shea (1984) introduces B-spline basis which are numerically stable. Fischer- Nychka- Zervos (1995) estimate the term structure using smoothing splines with roughness penalty where the penalty term is constant across the maturities but varying over time.

Although both methods provide reliable results, there are some differences between these approaches in terms of flexibility, smoothness and stability. Both methods tend to be quite flexible such that they are able to capture the movements in the yield curve. However, spline-based models can have much more flexibility as the number of maturity segments and the degree of the polynomial increases. The advantage of parametric models over spline-based models is that they are quite smooth due to the nature of their explicit functional form. To produce smooth yield curves, the curvature of forward rates are penalized in spline-based approaches. Lastly, in spline based models the different segments of the curve can move independently from each other. In other words, when a single point in data is changed, the nominal yield curve estimated through parametric methods can move substantially whereas the one estimated through spline based methods changes only slightly.

The motivation behind this paper is to examine whether spline based models with smoothing penalty functions provide better results for Turkish government term structure. For this purpose, it applies Cubic B-spline Basis with variable roughness penalty (VRP), employed by the Bank of England, to the Turkish bond market. It is the first comprehensive paper which applies this methodology to the Turkish bond market. As is discussed in more detail below, there are some empirical studies employing spline based methods for Turkish government bond data yet none of them uses any smoothing function. Besides, these studies are do not use recent data and do not cover the fixed coupon bonds, which establish the long end of the yield curve. Additionally, the paper compares in-sample fit and out-of-sample forecast of Cubic B-spline Basis with VRP to that of Extended Nelson-Siegel across different maturities.

2. Related Literature

Although there is a large number of empirical studies focusing on term structure modeling for developed countries, the studies on the construction of yield curve for Turkish bond market are relatively scarce. One of the studies about the term structure estimation methods for Turkish bond market is performed by Yoldas (2002). This study estimates the term structure through McCulloch cubic spline, Nelson-Siegel and Chambers-Carleton-Waldman exponential polynomial models using the end of month weighted closing prices of Turkish zero coupon bonds for the period from 1994 to 2002. The findings show that the goodness-of-fit of the exponential polynomial model is superior to others, especially on the long end of the yield curve. Alper, Akdemir and Kazimov (2004) study compares McCulloch cubic spline model and Nelson Siegel model using secondary government securities data from 1992 to 2004. They estimate monthly yield curves rather than daily yield curves using monthly volume weighted average of price and maturity and they exclude fixed coupon bonds due to their low liquidity. In sample and out-of-sample analysis shows that McCulloch has a better goodness-of-fit whereas Nelson-Siegel has better out-of-sample properties. The thesis written by Baki (2006) compares the performance of McCulloch and Nelson-Siegel model. He uses Turkish secondary government zero-coupon bond data for the period from January 2005 to June 2005. The results indicate that McCulloch's in-sample fit is better than that of

Nelson-Siegel. However, Nelson-Siegel tends to provide better fits for the maturities between 0-90 days whereas McCulloch's in-sample fit is better for all other time intervals.

Akinci et al. (2006) applies Nelson-Siegel and Extended Nelson-Siegel for Turkish bonds. They provide yield curve estimations using zero coupon and fixed coupon bonds for each day whereas previous studies use only zero coupon bonds. The inclusion of fixed coupon bonds allows precise estimation for zero rates at long maturities. It is stated in the paper that as the maturity structure of the bonds market improves over time, yield curve estimations are expected to be more precise. However, this study only focuses on construction of the yield curve with Extended Nelson- Siegel and does not compare the results with other models.

All studies about term structure of zero rates except Akinci et al. (2006) use only zero coupon bonds. This limits the yield curve estimations up to maximum two years maturity since there are no zero coupon bonds with maturity over two years. Additionally, these studies use weekly or monthly data and it does not allow policymakers to analyze the effect of macroeconomic events or policy events. This study allows comparison of Extended Nelson-Siegel and Cubic B-Spline Basis with VRP and it is the first comprehensive study comparing both methodologies with the data covering all zero coupon bonds and fixed coupon bonds.

3. Data and Methodology

3.1. Data

The data set for yield curve estimation covers zero coupon bonds and fixed coupon bonds and excludes floating rate bonds, sukuks and any other bonds with special characteristics. Since a fixed coupon bond is a linear combination of zero coupon bonds with different maturities, fixed coupon bonds are also used in the yield curve estimation. Additionally, bonds with different withholding tax rates are excluded so that each bond in the yield curve estimation has the same withholding tax rate.

Daily weighted average prices of the bonds in Borsa Istanbul (BIST) Debt Securities Market Daily Bulletins are used in the yield curve estimation. The bonds with forward value dates are excluded from the sample. The data set for the bonds covers the daily observations from February 2005 to August 2017. However, the data after January 2013 is used due to the insufficient number of long term bonds and spectrum of maturities.

3.2. Methodology

Since there is not sufficient amount of zero or coupon bonds to directly observe the yield curve, indirect methods have been used to estimate the yield curve. As an end product, these methods must produce either zero rates or forward rates for the whole maturity domain. The main objective in yield curve fitting is to minimize the weighted difference between the fitted prices and market prices of bonds. One of the common methods to prevent errors in short term zero rates is to weight the bonds with the inverse of their durations. Since the estimation errors for short term zero rates do not lead to substantial price differences, the shorter term bonds should have higher weights in the objective function.

$$min_{\beta} \ \sum_{i=1}^{N} \left(\frac{P_t^i - {P^{fit}}_t^i(\beta)}{D_t^i} \right)^2 \qquad \text{(I)}$$

where P_t^i and ${P^{fit}}_t^i(\beta)$ are actual and fitted prices of bond i. D_t^i represents the Macaulay duration of bond i. Fitted price is a function of parameters of β. The next section presents the most commonly used functions used in the yield curve estimation.

3.2.1. Extended Nelson Siegel

The most popular choice for parametric models is Extended Nelson-Siegel (ENS) model. The functional form of the instantaneous forward rate in Extended Nelson Siegel is defined as follows:

$$f(m,\beta) = \beta_0 + \beta_1 \, e^{-m/\tau_1} + \beta_2 \frac{m}{\tau_1} \, e^{-m/\tau_1} + \beta_3 \frac{m}{\tau_2} \, e^{-m/\tau_2} \qquad \text{(II)}$$

The parameters of ENS have economic meanings. β_0 gives the asymptote of zero coupon yield curve function and it is interpreted as the long run level of interest rates. β_1 reflects the deviation of the forward rates from the asymptotic level and it is mainly defined as the the difference between long term and short term instantaneous forward rates. β_2 and β_3 determine the magnitudes and the directions of humps. τ_1 and τ_2 represent the place of the hump. Determining the parameters of Extended Nelson-Siegel requires transforming instantaneous forward rates into zero rates to price coupon bearing bonds. Zero rates can be found out by integrating the instantaneous forward rates using equation II.

3.2.1. Cubic B-Spline Basis with Variance Roughness Penalty

The second model for Turkish bond market is the cubic B-spline basis methodology with VRP. In spline-based yield curve models, the maturity domain is partitioned into a number of segments and the points joining these segments are called knot points. For each of these segments, polynomials of the same degree are defined. Later on, constraints on function values and derivatives at these knot points are imposed for satisfying the continuity and differentiability.

B-spline is itself a polynomial spline (cubic polynomial in this application) which takes positive values for a number of adjacent segments (four segments in this case) and vanishes to zero for other segments. It can be written as follows:

$$B_{i,n}(m) = \frac{B_{i,n-1}(m)\,(m-k_i)}{(k_{i+n-1}-k_i)} + \frac{B_{i+1,n-1}(m)\,(k_{i+n}-m)}{(k_{i+n}-k_{i+1})} \quad \text{(III)}$$

for i = -3, -2, … ,(N-1) and n= 1,2,3,4. N+1 is the number of knot points. $B_{i,1}(m) = 1$ if τ is between k_i and k_{i+1} and 0 otherwise. The degree of the spline is shown by n and it is equal to 4 for cubic B-splines. The instantaneous forward rate, zero rate or price can be modeled as a linear function of the cubic B-splines. Instantaneous forward rates are modelled as following the literature.

$$f(m) = \sum_{i=1}^{N+3} \gamma_i \, B_{i,4}(m) \quad \text{(IV)}$$

Zero rates can be found by integrating the forward rate using equation IV.

Since spline-based modeling is a very flexible approach and both the number and location of the knot points are free to choose, it is easy to have a reasonable fit for any curve. On the other hand, this high flexibility has costs in terms of smoothness and stability. In order to increase numerical stability of piecewise polynomials, B-spline basis has been applied in yield curve studies. Furthermore, for having a reasonable level of smoothness, penalty functions are introduced. Penalty functions are designed to improve smoothness of the yield curves produced by spline-based techniques. When the penalty function is introduced, the objective function becomes:

$$min_{\beta} \sum_{i=1}^{N} \left(\frac{P_t^i - P{fit}_t^i(\beta)}{D_t^i} \right)^2 + \int_0^M \varphi(m)(f''(m))^2 \, dm \quad \text{(V)}$$

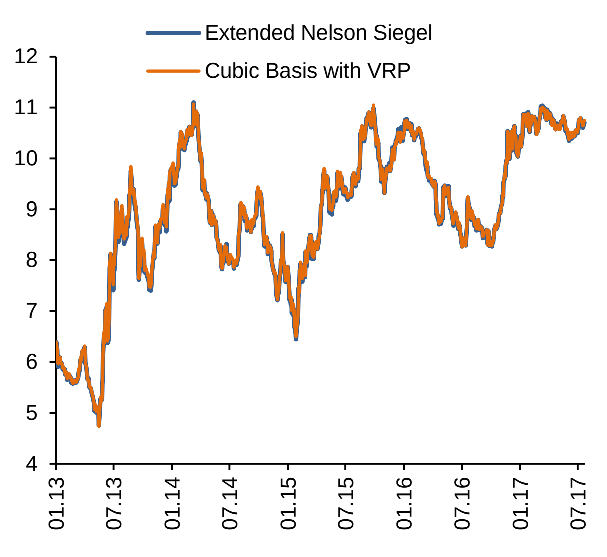
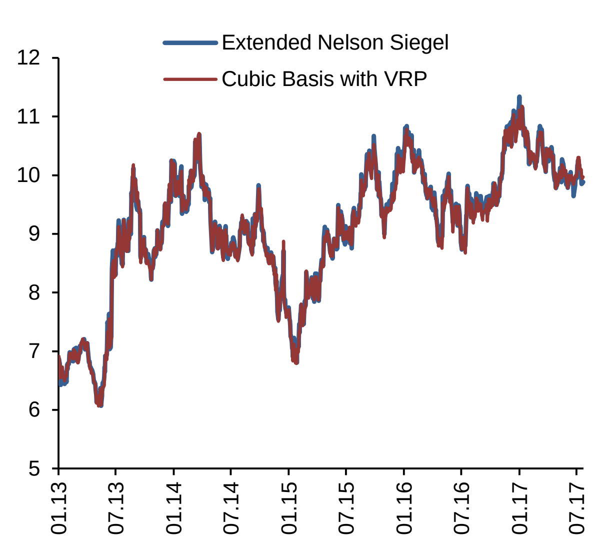
where m is the maturity, M is the longest maturity of the bonds, $f''(m)$ is the second derivative of the fitted forward rate curve (to preserve smoothness) and $\varphi(m)$ is the smoothing parameter. Specification of $\varphi(m)$can be done mainly in two ways, either by a constant optimal value (as proposed by Fisher et al. (1994)) throughout all maturity domain or by a variable roughness penalty which can vary across maturity (as introduced by Waggoner (1997)). Anderson and Sleath (2001) applies penalty term as a continuous function of three parameters.

$$log\ (\varphi(m)) = L - (L - S)e^{-m/\mu} \tag{VI}$$

L, S and μ are determined by maximizing the out-of-sample goodness-of-fit averaged over a sample period. In this note, Anderson and Sleath (2001) approach is used. A knot point at the maturity of every third bond is defined in order to define the knot sequence. Later the parameters of the penalty function are found out by comparing combinations of these parameters. It is found out that choosing L, S and μ as 10^4, $10^{0.5}$ and 3 respectively gives the best results among those combinations.

4. Empirical Findings

In this section, the empirical findings for the yield curve estimation are presented. Firstly, nominal zero rate yield curves are estimated using Treasury bonds and bills for the period from January 2013 to August 2017. Figure 1 and Figure 2 present the zero rate estimates for 2 years and 10 years using both VRP and ENS approaches. Both 2 year and 10 years estimates in both models tend to be quite close to each other. The figures show that Cubic B-Spline Basis (VRP) methodology can be confidently used as a backup for yield curve estimation.

Chart 1: Two Year Government Bond Yields	Chart 2: Ten Year Government Bond Yields
	
Source: BIST, Author's Calculations.	Source: BIST, Author's Calculations.

A further analysis is required to analyze the performances of both approaches. In this respect, firstly in-sample fit of both models are compared using in-sample errors. Two measures are used while comparing the estimation results. The first one is the root means square error (RMSE) which is defined as:

$$RMSE = \sqrt{\sum_{i=1}^{N} \frac{\left(P_t^i - Pfit_t^i\right)^2}{N}} \qquad \text{(VII)}$$

where N denotes the number of instruments used for the estimation. The second measure used is the mean absolute error (MAE) and it is defined as:

$$MAE = \sum_{i=1}^{N} \frac{|P_t^i - Pfit_t^i|}{N} \qquad \text{(VIII)}$$

The difference between these two measures is that RMSE gives more weight to larger errors. RMSE will always be larger than or equal to MAE. A higher difference between RMSE and MAE indicates that there is potentially a higher number of large errors. The RMSE and MAE statistics for the yield curve estimations are produced on a daily basis by using the in-sample pricing errors for each instrument available. Table 1 presents the mean, median, standard deviation and interquartile range of these statistics for the whole period.

Table 1: In Sample Pricing Errors					
Model	**Criteria**	**Mean**	**Median**	**Standard Deviation**	**3Q-1Q**
ENS	RMSE	0.160	0.151	0.066	0.076
	MAE	0.109	0.103	0.040	0.049
VRP	RMSE	0.155	0.147	0.061	0.061
	MAE	0.110	0.105	0.038	0.044

The estimation results are comparable for in-sample errors. VRP model produces on average 0.005 points lower RMSE and 0.001 points higher MAE. Additionally, in-sample pricing errors for different yield curve segments are presented in Table 2. For both models, in-sample errors get larger as the maturity becomes longer. This might be due the fact that the pricing errors are weighted with the inverse of the duration of the instruments. It is observed that for the maturities less than 2 years, ENS model is superior in terms of goodness of fit whereas for longer maturities VRP model gives better results.

Table 2: In Sample Pricing Errors Across Different Maturities						
Model	**Criteria**	**<1 Year**	**1-2 Years**	**2-5 Years**	**5-7 Years**	**7-10 Years**
ENS	RMSE	0.052	0.102	0.178	0.178	0.234
	MAE	0.037	0.085	0.146	0.171	0.203
VRP	RMSE	0.071	0.105	0.173	0.146	0.206
	MAE	0.056	0.089	0.140	0.139	0.179

While comparing different yield curve estimation methodologies, the predictive power of different methodologies should also be considered. Yield curves are desired to be robust to small changes in the underlying data. A common approach for evaluating the out of sample predictive power of the yield curves in the literature is to use out-of-sample pricing errors. In this respect, for each day and for each instrument, out-of-sample pricing error is calculated by using the derived yield curve of that day without that instrument. Later on, RMSE and MAE statistics are reproduced for these out-of-sample pricing errors on a daily basis.

Table 3 presents the mean, median, standard deviation and interquartile range of these statistics for the whole period. The results show that the performances of the two models in terms of stability are very much similar. This simply demonstrates that although ENS model has a parsimonious functional form which ensures a certain level of stability, VRP model can be compatible with ENS in terms of stability when the right penalty parameters are applied.

Table 3: Out-of Sample Pricing Errors					
Model	**Criteria**	**Mean**	**Median**	**Standard Deviation**	**3Q-1Q**
ENS	RMSE	0.229	0.207	0.107	0.106
	MAE	0.155	0.146	0.058	0.068
VRP	RMSE	0.221	0.204	0.106	0.100
	MAE	0.150	0.141	0.055	0.061

Table 4 presents out-of-sample pricing errors for different maturity segments. Similar to in-sample pricing errors, as the maturity gets longer, out-of-sample pricing errors get larger in general. ENS model performs best for maturity segment of 7 to 10 years whereas VRP model provides better out-of-sample fit for maturities shorter than 1 year.

Table 4: Out-of Sample Pricing Errors Across Different Maturities						
Model	**Criteria**	**<1 Year**	**1-2 Years**	**2-5 Years**	**5-7 Years**	**7-10 Years**
ENS	RMSE	0.145	0.116	0.216	0.211	0.329
	MAE	0.106	0.099	0.175	0.203	0.276
VRP	RMSE	0.087	0.127	0.221	0.214	0.348
	MAE	0.070	0.108	0.179	0.207	0.294

In the Turkish sovereign market, the number of instruments is smaller in the longer maturity segments. As a result, absence of an instrument means a relatively more loss of information while obtaining yield curve. Thus, out-of-sample pricing errors get larger as the maturities get longer. Performance of the models in terms of stability differs across maturities although the overall results are comparable. VRP is definitely superior for maturities up to 1 year whereas ENS model dominates for the remaining maturity segments. The outperformance of ENS over VRP might stem from the insufficient number of bonds for longer term maturities.

5. Conclusion

This paper employs Cubic B-Spline Basis with variable roughness penalty for Turkish government bonds using the data from January 2013 to August 2017. It is the first study that applies the spline based methodology with smoothing penalties using all available data and it compares the results with another popular yield curve estimation model, ENS. The paper firstly compares the zero rates at 2 and 10 years maturities estimated by Extended Nelson-Siegel and Cubic Basis Spline with VRP. The graphical findings indicate that both methods tend to provide quite similar results but there are some differences across different maturities.

In-sample fit and out-of-sample forecast for both ENS and VRP across different maturities are also presented. Although both models result in small errors, overall the VRP method provides a better in-sample fit than the ENS. It is also found that the VRP method provides a better in-sample fit for long-term zero rates. Out-of-sample results for the VRP are comparable with ENS but a further analysis shows that VRP's out-of sample forecast is better for the maturities up to 1 year whereas ENS's out-of-forecast results are better for the remaining maturities.

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